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On the uniqueness of nodal radial solutions of sublinear elliptic equations in a ball

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1. INTRODUCTION

We consider the second order ordinary differential equation

$$(1.1) \quad u'' + \frac{N-1}{r}u' + K(r)f(u) = 0, \quad 0 < r < 1,$$

with the boundary condition

$$(1.2) \quad u'(0) = u(1) = 0,$$

where $N \geq 2$, $K \in C^2[0, 1]$, $K(r) > 0$ for $0 \leq r \leq 1$, $f \in C^1(\mathbf{R})$, $sf(s) > 0$ for $s \neq 0$. Assume moreover that the following sublinear condition is satisfied:

$$(1.3) \quad \frac{f(s)}{s} > f'(s) \quad \text{for } s \neq 0.$$

Note that a solution of problem (1.1)–(1.2) is a radial solution $u(r)$ ($r = |x|$) of the Dirichlet problem of

$$\begin{cases} \Delta u + K(|x|)f(u) = 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where $B = \{x \in \mathbf{R}^N : |x| < 1\}$.

We consider solutions u of problem (1.1)–(1.2) satisfying $u(0) > 0$ only. If u is a solution of problem (1.1)–(1.2) with $u(0) < 0$, then it can be treated similarly as in the case where $u(0) > 0$, since $v \equiv -u$ satisfies $v(0) > 0$ and is a solution of

$$\begin{cases} v'' + \frac{N-1}{r}v' + K(r)f_0(v) = 0, & 0 < r < 1, \\ v'(0) = v(1) = 0, \end{cases}$$

where $f_0(s) = -f(-s)$.

In this paper we study the uniqueness of solutions of the problem (1.1)–(1.2) having exactly $k-1$ zeros in $(0, 1)$, where $k \in \mathbf{N}$.

Hence we consider the following problem:

$$(P_k) \quad \begin{cases} u'' + \frac{N-1}{r}u' + K(r)f(u) = 0, & 0 < r < 1, \\ u'(0) = u(1) = 0, & u(0) > 0, \\ u \text{ has exactly } k-1 \text{ zeros in } (0, 1). \end{cases}$$

It is known that there exists at least one solution of (P_k) under a certain condition. For example, in the case where $f(u) = |u|^{p-1}u$, $p > 0$, $p \neq 1$ and $N \geq 3$, the existence results of solutions of (P_k) were obtained by Y. Naito [4]. Assume that there exists limits f_0 and f_∞ such that

$$f_0 = \lim_{u \rightarrow 0} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \rightarrow \infty} \frac{f(u)}{u} \quad (0 \leq f_0, f_\infty \leq \infty).$$

In the case where there is a sufficiently large gap between f_0 and f_∞ , the existence of solutions of (P_k) was established by Dambrosio [1].

Now we consider the uniqueness of solutions of (P_k) . For the superlinear case $f(u) = |u|^{p-1}u$ ($p > 1$), Yanagida [6] showed that, for each $k \in \mathbb{N}$, (P_k) has at most one solution if $rK'(r)/K(r)$ is nonincreasing and $N \geq 3$. For the sublinear case where (3), $f_0 = \infty$ and $f_\infty = 0$, Kajikiya [2] proved that, for each $k \in \mathbb{N}$, the solution of (P_k) exists and is unique if $K(r) \equiv 1$. However very little is known about the uniqueness of solutions of (P_k) for the sublinear case and $K(r) \neq 1$.

The main result of this paper is as follows.

Theorem 1.1. *Suppose that (1.3) holds. If*

$$(1.4) \quad 3r^2(K')^2 - 2r^2KK'' + 2(N-1)rKK' + 4(N-1)K^2 \geq 0, \quad 0 \leq r \leq 1,$$

then, for each $k \in \mathbb{N}$, (P_k) has at most one solution.

In view of the following equality

$$\begin{aligned} & 3r^2(K')^2 - 2r^2KK'' + 2(N-1)rKK' + 4(N-1)K^2 \\ &= K^2 \left[\left(\frac{rK'}{K} + 2 \right) \left(\frac{rK'}{K} + 2(N-1) \right) - 2r \left(\frac{rK'}{K} \right)' \right], \end{aligned}$$

we have the following corollary of Theorem 1.1.

Corollary 1.1. *Suppose that (1.3) holds. Assume moreover that one of the following (1.5)–(1.7) is satisfied:*

$$(1.5) \quad K'' \leq 0, \quad K' \geq 0 \quad \text{for } 0 \leq r \leq 1,$$

$$(1.6) \quad N = 2, \quad \left(\frac{rK'}{K} \right)' \leq 0 \quad \text{for } 0 \leq r \leq 1,$$

$$(1.7) \quad N > 2, \quad \frac{rK'}{K} \geq -2, \quad \left(\frac{rK'}{K} \right)' \leq 0 \quad \text{for } 0 \leq r \leq 1.$$

Then, for each $k \in \mathbb{N}$, (P_k) has at most one solution.

2. LEMMAS

In this section we give several lemmas.

First we note that (1.1) can be rewritten as follows:

$$(2.1) \quad (r^{N-1}u')' + r^{N-1}K(r)f(u) = 0, \quad 0 < r < 1.$$

The proof of Theorem 1.1 is based on the method of Kolodner [3]. Namely we consider the solution $u(r, \alpha)$ of (1.1) satisfying the initial condition

$$(2.2) \quad u(0) = \alpha > 0, \quad u'(0) = 0,$$

where $\alpha > 0$ is a parameter. Since $K \in C^2[0, 1]$ and $f \in C^1(\mathbf{R})$, we see that $u(r, \alpha)$ exists on $[0, 1]$ is unique and satisfies $u, u' \in C^1([0, 1] \times (0, \infty))$, and that $u_\alpha(r, \alpha)$ is a solution of linearized problem

$$(2.3) \quad \begin{cases} (r^{N-1}w')' + r^{N-1}K(r)f'(u(r, \alpha))w = 0, & r \in (0, 1], \\ w(0) = 1, \quad w'(0) = 0. \end{cases}$$

(See, for example, [5, §6 and 13].)

Hereafter we assume that $u(r, \alpha)$ is a solution of (P_k) . Let z_i be the i -th zero of $u(r, \alpha)$. Let $t_1 = 0$. For each $i \in \{2, 3, \dots, k\}$, there exists $t_i \in (z_{i-1}, z_i)$ such that $u'(t_i, \alpha) = 0$, since $u(r, \alpha)(r^{N-1}u'(r, \alpha))' < 0$ for $r \in (z_i, z_{i+1})$. Therefore we find that

$$0 = t_1 < z_1 < t_2 < z_2 < \dots < t_{k-1} < z_{k-1} < t_k < z_k = 1,$$

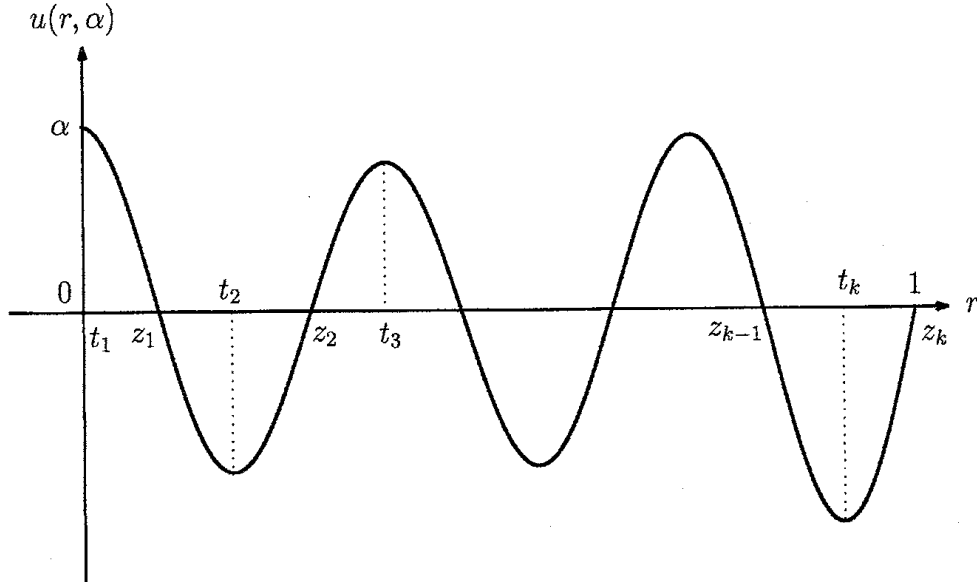
$$u(z_i, \alpha) = 0, \quad u'(t_i, \alpha) = 0, \quad i = 1, 2, \dots, k,$$

$$u(r, \alpha) > 0 \quad \text{for } r \in [t_1, z_1),$$

$$(2.4) \quad (-1)^i u(r, \alpha) > 0 \quad \text{for } r \in (z_i, z_{i+1}), \quad i = 1, 2, \dots, k-1,$$

$$(2.5) \quad (-1)^i u'(r, \alpha) > 0 \quad \text{for } r \in (t_i, t_{i+1}), \quad i = 1, 2, \dots, k-1,$$

$$(2.6) \quad (-1)^k u'(r, \alpha) > 0 \quad \text{for } r \in (t_k, z_k].$$



Lemma 2.1. Assume that (1.3) holds. Let w be the solution of (2.3). Then $w(r) > 0$ for $x \in [0, z_1]$.

Proof. Note that $w(0) = 1$ and $w'(0) = 0$. Assume to the contrary that there exists a number $r_1 \in (0, z_1]$ such that $w(r) > 0$ for $r \in [0, r_1)$ and $w(r_1) = 0$. Then

we see that $w'(r_1) < 0$. Let $u \equiv u(r, \alpha)$. An easy computation shows that

$$(2.7) \quad [r^{N-1}(w'u - uw')] = r^{N-1}K(r)[f(u) - f'(u)u]w.$$

Recall that $u(r) > 0$ for $r \in [0, z_1]$. Integrating of (2.7) over $[0, r_1]$ and using (1.3), we have

$$r_1^{N-1}w'(r_1)u(r_1) = \int_0^{r_1} r^{N-1}K(r)[f(u) - f'(u)u]w \, dr > 0,$$

which implies $w'(r_1) > 0$. This is a contradiction. Consequently we find that $w(r) > 0$ for $r \in (0, z_1]$.

Lemma 2.2. *Assume that (1.3) holds. For each $i \in \{1, 2, \dots, k-1\}$, the solution w of (2.3) has at most one zero in $[z_i, z_{i+1}]$.*

Proof. Note that $u \equiv u(r, \alpha)$ is a solution of

$$(r^{N-1}u')' + r^{N-1}K(r)\frac{f(u)}{u}u = 0, \quad r \in (z_i, z_{i+1})$$

and satisfies $u(z_i) = u(z_{i+1}) = 0$ and $u(r) \neq 0$ for $r \in (z_i, z_{i+1})$. From (1.3) it follows that

$$r^{N-1}K(r)f'(u) < r^{N-1}K(r)\frac{f(u)}{u}, \quad r \in (z_i, z_{i+1}).$$

Assume to the contrary that there exist numbers r_0 and r_1 such that $z_i \leq r_0 < r_1 \leq z_{i+1}$ and $w(r_0) = w(r_1) = 0$. Then Sturm's comparison theorem implies that u has at least one zero in (r_0, r_1) . This is a contradiction. The proof is complete.

The following identity plays a crucial part in the proof of Theorem 1.1.

Lemma 2.3. *Let $u \equiv u(r, \alpha)$ and let w be the solution of (2.3). Then*

$$(2.8) \quad \begin{aligned} & \left[r^{N-1}K^{-\frac{1}{2}}[w'u' - wu''] - r^{N-1}(K^{-\frac{1}{2}})'wu' \right]' \\ &= -\frac{r^{N-2}}{4K^{\frac{5}{2}}} \left[3r^2(K')^2 - 2r^2KK'' + 2(N-1)rKK' + 4(N-1)K^2 \right] w \frac{u'}{r}. \end{aligned}$$

for $0 < r \leq 1$.

Proof. A direct calculation shows that (2.8) follows immediately.

Remark 2.1. We note that

$$(2.9) \quad u''(0, \alpha) = \lim_{r \rightarrow +0} \frac{u'(r, \alpha)}{r} = -\frac{K(0)f(\alpha)}{N},$$

and hence, the right side of (2.8) is continuous for $0 \leq r \leq 1$. In fact, by integrating (2.1) over $[0, r]$, we see that

$$u'(r, \alpha) = -r^{-(N-1)} \int_0^r t^{N-1}K(t)f(u(t, \alpha))dt, \quad r \in [0, 1],$$

so that

$$-\frac{r}{N} \max_{t \in [0, r]} K(t)f(u(t, \alpha)) \leq u'(r, \alpha) \leq -\frac{r}{N} \min_{t \in [0, r]} K(t)f(u(t, \alpha)), \quad r \in [0, 1].$$

Then we obtain (2.9).

Lemma 2.4. Assume that (1.4) holds. Then the solution w of (2.3) has at least one zero in $(t_i, t_{i+1}]$ for each $i \in \{1, 2, \dots, k-1\}$.

Proof. Suppose that $w(r) \neq 0$ for $r \in (t_i, t_{i+1}]$. We may assume that $w(r) > 0$ for $r \in (t_i, t_{i+1}]$, since the case where $w(r) < 0$ for $r \in (t_i, t_{i+1}]$ can be treated similarly. Then we have $w(t_i) \geq 0$, $w(t_{i+1}) > 0$. In view of (1.1) we have

$$u''(t_j) = -K(t_j)f(u(t_j)), \quad j = 2, 3, \dots, k.$$

From (2.4) and (2.9) it follows that $(-1)^j u''(t_j) > 0$ for $j = 1, 2, \dots, k$. Consequently we have

$$(-1)^i (-g(t_{i+1})w(t_{i+1})u''(t_{i+1}) + g(t_i)w(t_i)u''(t_i)) > 0,$$

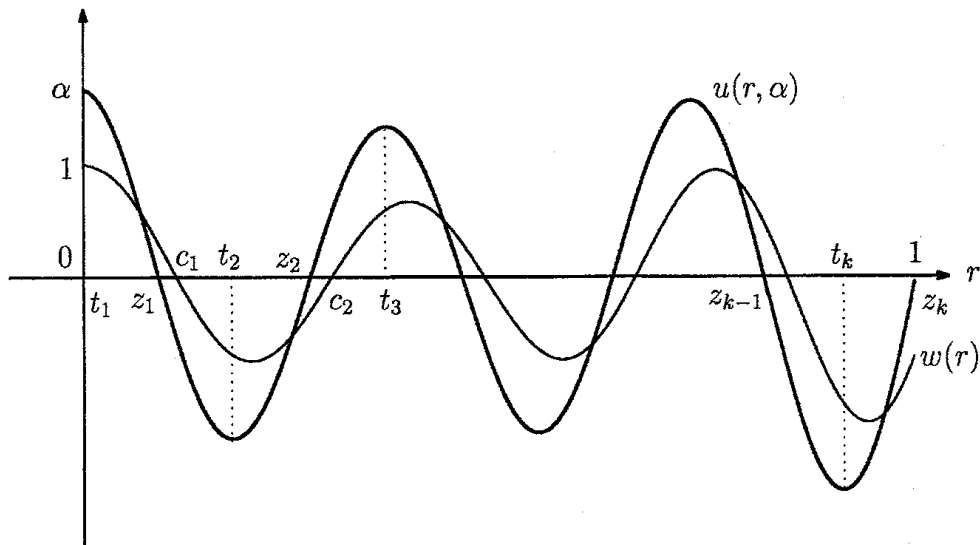
where $g(r) = r^{N-1}[K(r)]^{-\frac{1}{2}}$. On the other hand, integrating (2.8) over $[t_i, t_{i+1}]$ and using (1.4) and (2.5), we find that

$$(-1)^i (-g(t_{i+1})w(t_{i+1})u''(t_{i+1}) + g(t_i)w(t_i)u''(t_i)) \leq 0.$$

This is a contradiction. The proof is complete.

Lemma 2.5. Let w be the solution of (2.3). Assume that (1.3) and (1.4) hold. Then $(-1)^i w(z_i) < 0$ for $i = 1, 2, \dots, k$.

Proof. Lemma 2.1 implies that $w(z_1) > 0$. By Lemmas 2.1 and 2.4, there exists a number $c_1 \in (z_1, t_2]$ such that $w(r) > 0$ for $r \in [0, c_1]$ and $w(c_1) = 0$. Then Lemma 2.2 implies that $w(r) < 0$ for $r \in (c_1, z_2]$. Hence we have $w(z_2) < 0$. From Lemma 2.4 it follows that there exists a number $c_2 \in (z_2, t_3]$ such that $w(r) < 0$ for $r \in (c_1, c_2)$ and $w(c_2) = 0$. By Lemma 2.2 we see that $w(r) > 0$ for $r \in (c_2, z_3]$, so that $w(z_3) > 0$. By continuing this process, we conclude that $(-1)^i w(z_i) < 0$ for $i = 1, 2, \dots, k$. The proof is complete.



3. PROOF OF THEOREM 1.1

In this section we give the proof of Theorem 1.1. To this end we employ the Prüfer transformation for the solution $u(r, \alpha)$ of problem (1.1)–(2.2). For the solution $u(r, \alpha)$ with $\alpha > 0$, we define the functions $\rho(r, \alpha)$ and $\theta(r, \alpha)$ by

$$\begin{aligned} u(r, \alpha) &= \rho(r, \alpha) \sin \theta(r, \alpha), \\ r^{N-1} u'(r, \alpha) &= \rho(r, \alpha) \cos \theta(r, \alpha), \end{aligned}$$

where $' = d/dx$. Since $u(r, \alpha)$ and $u'(r, \alpha)$ cannot vanish simultaneously, $\rho(r, \alpha)$ and $\theta(r, \alpha)$ are written in the forms

$$\rho(r, \alpha) = ([u(r, \alpha)]^2 + r^{2(N-1)}[u'(r, \alpha)]^2)^{\frac{1}{2}} > 0$$

and

$$\theta(r, \alpha) = \arctan \frac{u(r, \alpha)}{r^{N-1} u'(r, \alpha)},$$

respectively. Therefore, since $u, u' \in C^1([0, 1] \times (0, \infty))$, we find that $\rho, \theta \in C^1([0, 1] \times (0, \infty))$. From the initial condition (2.2) it follows that $\rho(0, \alpha) = \alpha$ and $\theta(0, \alpha) \equiv \pi/2 \pmod{2\pi}$. For simplicity we take $\theta(0, \alpha) = \pi/2$. By a simple calculation we see that

$$\theta'(r, \alpha) = \frac{1}{r^{N-1}} \cos^2 \theta(r, \alpha) + r^{N-1} K(r) \frac{\sin \theta(r, \alpha) f(\rho(r, \alpha) \sin \theta(r, \alpha))}{\rho(r, \alpha)} > 0$$

for $r \in (0, 1]$, which shows that $\theta(r, \alpha)$ is strictly increasing in $r \in (0, 1]$ for each fixed $\alpha > 0$. It is easy to see that $u(r, \alpha)$ is a solution of (P_k) if and only if

$$(3.1) \quad \theta(1, \alpha) = k\pi,$$

Hence the number of solutions of (P_k) is equal to the number of roots $\alpha > 0$ of (3.1).

Proposition 3.1. *Let $k \in \mathbb{N}$ and let $u(r, \alpha_0)$ be a solution of (P_k) for some $\alpha_0 > 0$. Suppose that (1.3) and (1.4) hold. Then $\theta_\alpha(1, \alpha_0) < 0$.*

Proof. Observe that

$$\theta_\alpha(r, \alpha) = \frac{u_\alpha(r, \alpha) r^{N-1} u'(r, \alpha) - u(r, \alpha) r^{N-1} u'_\alpha(r, \alpha)}{[u(r, \alpha)]^2 + [u'(r, \alpha)]^2}.$$

Since $u(1, \alpha_0) = 0$ and $z_k = 1$, we obtain

$$\theta_\alpha(1, \alpha_0) = \frac{u_\alpha(z_k, \alpha_0)}{u'(z_k, \alpha_0)}.$$

Note that $(-1)^k u'(z_k, \alpha_0) > 0$, because of (2.6). From Lemma 2.5, it follows that $(-1)^k u_\alpha(z_k, \alpha_0) < 0$, which implies that $\theta_\alpha(1, \alpha_0) < 0$. The proof is complete.

Proof of Theorem 1.1. Assume to the contrary that there exist numbers $\alpha_1 > 0$ and $\alpha_2 > 0$ such that $u(r, \alpha_1)$ and $u(r, \alpha_2)$ are solutions of (P_k) and $\alpha_1 \neq \alpha_2$. Then $\theta(1, \alpha_1) = \theta(1, \alpha_2) = k\pi$. We may assume without loss of generality that $0 < \alpha_1 < \alpha_2$ and $\theta(1, \alpha) \neq k\pi$ for $\alpha \in (\alpha_1, \alpha_2)$. In view of Proposition 3.1, we

conclude that $\theta_\alpha(1, \alpha_1) < 0$ and $\theta_\alpha(1, \alpha_2) < 0$. The intermediate value theorem implies that there is a number $\alpha_0 \in (\alpha_1, \alpha_2)$ such that $\theta(1, \alpha_0) = k\pi$. This is a contradiction. Consequently, (P_k) has at most one solution. The proof of Theorem 1.1 is complete.

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